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On Sieved Orthogonal Polynomials. VIII. Sieved Associated Pollaczek Polynomials

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We study a general orthogonal polynomial set which includes the sieved associated ultraspherical and the sieved Pollaczek polynomials. This we get by letting q approach a root of unity in the recurrence relation and the generating functions of the associated q-ultraspherical and the Pollaczek polynomials. We find the weight functions with respect to which these polynomials are orthogonal and determine the asymptotic behavior of these polynomials on and off their interval of orthogonality. © 1992 Academic Press, Inc.

1. INTRODUCTION

Al-Salam, Allaway, and Askey [1] introduced two interesting families of orthogonal polynomials which they called sieved ultraspherical polynomials of the first and second kinds. To get them they let q tend to a root of unity in the three term recurrence relation for the continuous q-ultraspherical polynomials, $\{C_n(x; \beta \mid q)\}$, which may be defined [2] by

$$C_{-1}(x; \beta \mid q) = 0, \qquad C_0(x; \beta \mid q) = 1, \tag{1.1}$$

and, for $n > 0$, by

$$\begin{aligned} 2x(1 - \beta q^n)\, C_n(x; \beta \mid q) &= (1 - q^{n+1})\, C_{n+1}(x; \beta \mid q) \\ &\quad + (1 - \beta^2 q^{n-1})\, C_{n-1}(x; \beta \mid q). \end{aligned} \tag{1.2}$$

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More specifically Al-Salam, Allaway, and Askey let $\omega_k = \exp\{2\pi i/k\}$, k being a positive integer. Then they showed that the sieved ultraspherical polynomials of the first and second kinds, which they defined, respectively, by

$$c_n^\lambda(x; k) = \lim_{s \to 1} \frac{(s\omega_k; s\omega_k)}{(s^{2\lambda k}; s\omega_k)_n} C_n(x; s^{k\lambda} \mid s\omega_k)$$

$$B_n^\lambda(x; k) = \lim_{s \to 1} C_n(x; s^{k\lambda+1}\omega_k \mid s\omega_k),$$

may be generated by the recurrence relations

$$\begin{gathered} c_0^\lambda(x; k) = 1, \qquad c_1^\lambda(x; k) = x \\ c_{n+1}^\lambda(x; k) = 2x c_n^\lambda(x; k) - c_{n-1}^\lambda(x; k) \qquad (\text{if } k \nmid n), \qquad (1.3) \\ (m + 2\lambda)\, c_{mk+1}^\lambda(x; k) = 2x(m + \lambda)\, c_{mk}^\lambda(x; k) - m c_{mk-1}^\lambda(x; k), \end{gathered}$$

and

$$\begin{aligned} &B_0^\lambda(x; k) = 1, \qquad B_1^\lambda(x; k) = 2x, \\ &B_{n+1}^\lambda(x; k) = 2x B_n^\lambda(x; k) - B_{n-1}^\lambda(x; k) \qquad (\text{if } k \nmid n + 1), \\ &m B_{mk}^\lambda(x; k) = 2x(m + k)\, B_{mk-1}^\lambda(x; k) \qquad\qquad (1.4) \\ &\qquad - (m + 2\lambda)\, B_{mk-2}^\lambda(x; k). \end{aligned}$$

The work in [1] generated a great deal of interest and led to various generalizations and applications. (See [4–9, 11, 12].)

Interesting applications of sieved polynomials have been given by Van Assche and Magnus [16]. They used them to prove that a discrete system of orthogonal polynomials introduced by Lubinsky [14] has convergent recurrence coefficients. In addition Van Assche and Magnus [16] used sieved polynomials to construct two examples of polynomial sets which are orthogonal with respect to discrete measures with masses that are dense in the interval $[-1, 1]$ and proved that the coefficients in the three term recurrence relations of the corresponding orthonormal systems converge (as $n \to \infty$). These two examples were easier to handle than Lubinsky's original examples which show that a certain sufficient condition of Rahmanov is not necessary.

The sieved ultraspherical polynomials also turned out to be spherical harmonics in $\mathscr{C}^n$ where the corresponding Laplacian is modified to include terms corresponding to reflections (see [9]).

In [12] symmetric sieved Pollaczak polynomials were introduced by sieving the q-Pollaczek polynomials instead of sieving the continuous q-ultraspherical polynomials. The general sieved Pollaczek polynomials were introduced and studied in some details in [7].

The purpose of this work is to further extend the work on sieved Pollaczek polynomials. Our extension contains two additional parameters. One parameter corresponds to replacing n by $n+c$ in the coefficients of the three term recurrence relation of the general sieved Pollaczek polynomials. The second replaces the divisibility of n and $n+1$ by k in (1.3) and (1.4) by divisibility of $n+r$ by k where r is a non-negative integer. The parameter c is an association parameter. In particular special cases of our work contain generalizations of sieved ultraspherical polynomials in which $k \nmid n$ or $k \nmid n+1$ are replaced by $k \nmid n+r$. We feel that the degree of freedom gained by introducing the parameter r is useful.

The paper is arranged as follows. In Section 2 we define the sieved associated Pollaczek polynomial of the second kind by sieving the recurrence relations of the associated q-Pollaczek polynomials. We also apply the same sieving process to generating functions of the associated q-Pollaczek polynomials and derive a generating function for the polynomials under consideration. In Section 3 the asymptotics of the sieved associated polynomials and their corresponding numerator polynomials are derived. In Section 4 the Hadamard singular integral is used to find the absolutely continuous component of the measure with respect to which our polynomials are orthogonal. For the definition and various properties of the Hadamard integral see [3, 7]. In Section 5 we discuss the nature of the discrete spectrum and mention some open problems.

We shall make use of the Darboux method for finding asymptotic expansions of the coefficients of certain generating functions which, in our case, happen to be orthogonal polynomials. This method is explained in [3].

It may be of interest to note that the results of this work do not follow from the general theories of sieved polynomials in [6, 11]. The reason is that [6] deals only with the symmetric polynomials while the result [11] do not even include the sieved Pollaczak polynomials (the special case $c=r=0$ of our work) as it was pointed out in [8].

The sieved polynomials of the first kind can be treated in the same way.

2. Sieved Associated Pollaczek Polynomials

In this section we study the sieved associated Pollaczek polynomials of the second kind and their generating functions.

Charris and Ismail [7] started with an analog of the continuous q-ultraspherical polynomials, namely,

$$\begin{aligned}(1-q^{n+1})\,F_{n+1}(x) &= 2[(1-U\Delta q^n)x+Vq^n]\,F_n(x)\\ &\quad -(1-\Delta^2 q^{n-1})\,F_{n-1}(x), \qquad n>0 \qquad (2.1)\\ F_0(x)=1, &\qquad F_{-1}(x)=0.\end{aligned}$$

One can then define [13] the associated q-Pollaczek polynomials by

$$\begin{aligned}(1-\gamma q^{n+1})\,F^{(\gamma)}_{n+1}(x) &= 2[(1-\gamma U\Delta q^n)x+\gamma Vq^n]\,F^{(\gamma)}_n(x)\\ &\quad -(1-\gamma\,\Delta^2 q^{n-1})\,F^{(\gamma)}_{n-1}(x), \qquad (2.2)\\ F^{(\gamma)}_0(x)=1, &\qquad F^{(\gamma)}_{-1}(x)=0.\end{aligned}$$

We shall use the notation $F^{(\gamma)}_n(x; U, V, \Delta; q)$ instead of $F^{(\gamma)}_n(x)$ if we need to show the dependence on the parameters γ, U, Δ, V, and q.

Now let r be a non-negative integer, $0 \leqslant r < k$, a, b, c real numbers, $\lambda > -1/2$, and $|q| < 1$. Set in (2.2)

$$\begin{aligned}&q=\omega s, \qquad \omega=\exp(2\pi i/k),\\ &\gamma = s^{kc+r}\omega^r, \qquad (2.3)\\ &U=s^{ka}, \qquad V=\omega s^k(1-s^{kb}), \qquad \Delta=\omega s^{k\lambda+1}.\end{aligned}$$

Definition. The sieved associated Pollaczek polynomials of the second kind are defined as

$$Q^{(\lambda,r)}_n(x)=Q^{(\lambda,r)}_n(x;a,b,c;k)=\lim_{s\to 1} F^{\gamma}_n(x;U,V,\Delta;q), \qquad (2.4)$$

where γ, U, V, Δ, and q are set as above in (2.3).

Thus (2.2) establishes the following theorem:

Theorem 1. *The polynomials $Q^{(\lambda,r)}_n(x)$ can be generated by the initial conditions*

$$\begin{aligned}&Q^{(\lambda,r)}_0(x)=1,\\ &Q^{(\lambda,r)}_1(x)=2x \qquad \text{if} \quad r\neq k-1 \qquad (2.5)\\ &\qquad =2[(\lambda+a+c+1)x+b]/(c+1) \qquad \text{if} \quad r=k-1,\end{aligned}$$

and the recurrence relation

$$\begin{aligned} Q_{n+1}^{(\lambda,r)}(x) &= 2xQ_n^{(\lambda,r)}(x) - Q_{n-1}^{(\lambda,r)}(x), \qquad \text{if} \quad k \nmid n+r+1 \\ (c+m)\, Q_{mk-r}^{(\lambda,r)}(x) &= 2[(\lambda+a+c+m)x+b]\, Q_{mk-r-1}^{(\lambda,r)}(x) \\ &\quad - (2\lambda+m+c)\, Q_{mk-r-2}^{(\lambda,r)}(x), \qquad \text{otherwise}. \end{aligned} \tag{2.6}$$

We now determine the generating function of $\{F_n^{(\gamma)}(x)\}$, namely,

$$F^{(\gamma)}(x, t) = \sum_{n=0}^{\infty} F_n^{(\gamma)}(x) t^n. \tag{2.7}$$

Multiplying (2.2) by t^{n+1} and summing for $n = 1, 2, 3, \ldots$ we get the q-difference equation

$$(1 - 2xt + t^2)\, F^{(\gamma)}(x, t) = (1-\gamma) + \gamma[\Delta^2 t^2 - 2(xU\Delta - V)t + 1]\, F^{(\gamma)}(x, qt). \tag{2.8}$$

Now put $t^2 - 2xt + 1 = (1 - t/\alpha)(1 - t/\beta)$ and $\Delta^2 t^2 - 2(xU\Delta - V)t + 1 = (1 - t/\xi)(1 - t/\zeta)$. Thus (2.8) implies that

$$\begin{aligned} F^{(\gamma)}(x, t) &= (1-\gamma) \sum_{n=0}^{\infty} \frac{(t/\xi, q)_n\, (t/\zeta, q)_n}{(t/\alpha, q)_{n+1}\, (t/\beta, q)_{n+1}}\, \gamma^n \\ &= \frac{1-\gamma}{(1-t/\alpha)(1-t/\beta)}\, {}_3\Phi_2\left(\begin{matrix} t/\xi,\ t/\zeta,\ q;\ \gamma \\ tq/\alpha,\ tq/\beta \end{matrix}\right) \\ &= F(x, t) \sum_{m, j \geqslant 0} \frac{(\xi q^2/\alpha, q)_j\, (\xi q^2/\beta, q)_m}{(q, q)_j\, (q, q)_m} \\ &\quad \cdot \left(\frac{t}{\alpha}\right)^j \left(\frac{\alpha}{\xi}\right)^j \left(\frac{t}{\beta}\right)^m \left(\frac{\beta}{\zeta}\right)^m \\ &\quad \cdot \left(\frac{1 - \xi q/\alpha}{1 - \xi q^{j+1}/\alpha}\right) \left(\frac{1 - \zeta q/\beta}{1 - \zeta q^{m+1}/\beta}\right) \left(\frac{1-\gamma}{1 - \gamma q^{m+j}}\right), \end{aligned} \tag{2.9}$$

where

$$F(x, t) = F^{(1)}(x, t) = \frac{(t/\xi, q)_\infty\, (t/\zeta, q)_\infty}{(t/\alpha, q)_\infty\, (t/\beta, q)_\infty}$$

is the generating function for the q-Pollaczek polynomials [7].

We now perform the sieving process (2.3) and (2.4) on (2.9) to get the generating function for the sieved associated Pollaczek polynomials of the second kind. We get

$$Q^{(\lambda,r)}(x,t)=\frac{1}{(1-t/\alpha)(1-t/\beta)}+\frac{(-B)(\alpha t)^{k-r}}{(1-t/\alpha)(1-t/\beta)}$$

$$\cdot\left(1-\frac{t^k}{\alpha^k}\right)^A\left(1-\frac{t^k}{\beta^k}\right)^B$$

$$\cdot\int_0^1 u^c\left(1-u\frac{t^k}{\alpha^k}\right)^{-A}\left(1-u\frac{t^k}{\beta^k}\right)^{-B-1}du$$

$$+\frac{(-A)(\beta t)^{k-r}}{(1-t/\alpha)(1-t/\beta)}\left(1-\frac{t^k}{\alpha^k}\right)^A\left(1-\frac{t^k}{\beta^k}\right)^B$$

$$\cdot\int_0^1 u^c\left(1-u\frac{t^k}{\beta^k}\right)^{-B}\left(1-u\frac{t^k}{\alpha^k}\right)^{-A-1}du, \qquad (2.10)$$

where $A=A(z)=-\lambda+(az+b)/\sqrt{z^2-1}$, $B=B(z)=-\lambda-(az+b)/\sqrt{z^2-1}$, $\alpha:=\alpha(z)=z+\sqrt{z^2-1}$, $\beta:=z-\sqrt{z^2-1}$, and $z\notin\mathscr{C}/[-1,1]$. For detailed calculation of the above sieving process see [7].

The case $k=1$, $r=0$ gives the generating function for the associated Pollaczek polynomials [7]. On the other hand if $a=b=r=0$ and $k=1$ we get the generating function for the associated ultraspherical polynomials [5].

We now simplify (2.10) to get

$$Q^{(\lambda,r)}(x,t)=\frac{t^r-\beta^r}{t^r(t-\alpha)(t-\beta)}+A(\beta^{k+r}-\beta^{k-r})t^{k-r}$$

$$\cdot B^\lambda(x,t)\int_0^1\left(1-u\frac{t^k}{\alpha^k}\right)^{-A-1}\left(1-u\frac{t^k}{\beta^k}\right)^{-B}u^c\,du$$

$$+c(\alpha t)^{-r}B^\lambda(x,t)\int_0^1\left(1-u\frac{t^k}{\alpha^k}\right)^{-A}$$

$$\cdot\left(1-u\frac{t^k}{\beta^k}\right)^{-B}u^{c-1}\,du, \qquad (2.11)$$

where $B^\lambda(x,t)=(1-2xt+t^2)^{-1}(1-t^k/\alpha^k)^A(1-t^k/\beta^k)^B$ is the generating function for the sieved Pollaczek polynomials of the second kind [7].

If we put $r=1$ and take the limit as $c \to 0$ we get

$$B^{*\lambda}(x,t) = 2tQ^{(\lambda,r)}(x,t)$$
$$= \frac{2}{t-\alpha} + 2B^{\lambda}(x,t)\left[(\beta-\alpha)\left(\frac{t}{\alpha}\right)^k \cdot A\int_0^1 \left(1-u\frac{t^k}{\alpha^k}\right)^{-A-1}\left(1-u\frac{t^k}{\beta^k}\right)^{-B} du + \beta\right],$$

which is Eq. (3.27) of [7] giving the generating function for the numerator polynomials associated with the sieved Pollaczek polynomials of the second kind.

Another family of generating functions can be derived by using the following lemma [12]

LEMMA 1. *If $P(t) = \sum_{n=0}^{\infty} p_n t^n$, then*

$$\sum_{j=0}^{k-1} P(t\omega^j)\omega^{-lj} = k \sum_{n=0}^{\infty} p_{nk+l} t^{nk+l},$$

for $l = 0, 1, 2, ..., k-1$.

Using Lemma 1 and formula (2.10) we get the following theorem.

THEOREM 2. *The sections, $G_l^{(r)}(x,t)$, of the generating function $Q^{(\lambda,r)}(x,t)$ are given by*

$$G_l^{(r)}(x,t) := \sum_{n=0}^{\infty} Q_{nk+l}^{(\lambda,r)}(x) t^n$$
$$= \frac{1}{(1-t/\alpha^k)(1-t/\beta^k)} (U_l(x) - U_{l+r}(x))$$
$$+ \frac{t}{(1-t/\alpha^k)(1-t/\beta^k)} (U_{k-l-2}(x) - U_{k-l-r-2}(x))$$
$$+ (U_{l+r}(x) + tU_{k-l-r-2}(x)) H(x,t), \tag{2.12}$$

where $U_j(x)$ is the Chebyshev polynomial of the second kind and

$$H(x, t) = \frac{1}{(1 - t/\alpha^k)(1 - t/\beta^k)} - t\left(1 - \frac{t}{\alpha^k}\right)^{A-1}\left(1 - \frac{t}{\beta^k}\right)^{B-1}$$

$$\cdot\left[\alpha^{k-r} B \int_0^1 \left(1 - v\frac{t}{\alpha^k}\right)^{-A}\left(1 - v\frac{t}{\beta^k}\right)^{-B-1} v^c\, dv\right.$$

$$\left.+ \beta^{k-r} A \int_0^1 \left(1 - v\frac{t}{\beta^k}\right)^{-B}\left(1 - v\frac{t}{\alpha^k}\right)^{-A-1} v^c\, dv\right],$$

where $l, r = 0, 1, 2, ..., k-1$.

The proof follows from (2.10), the lemma, and the relations

$$\sum_{n=0}^{\infty} U_{l+kn}(x)\, t^n = \frac{1}{(1 - t/\alpha^k)(1 - t/\beta^k)}\,(U_l(x) + tU_{k-l-2}(x)),$$

$$\sum_{n=0}^{\infty} U_{l+r+kn}(x)\, t^n = \frac{1}{(1 - t/\alpha^k)(1 - t/\beta^k)}\,(U_{l+r}(x) + tU_{k-l-r-2}(x)).$$

Put $r = 0$, $k = 1$ in (2.12) and we clearly get the generating function of the associated Pollaczek polynomials [7],

$$g(x, t) = \sum_{n=0}^{\infty} P_n^{(\lambda+1)}(\cos\theta, a, b, c, 1)\, t^n$$

$$= \frac{1}{(1 - t/\alpha)(1 - t/\beta)} - t\left(1 - \frac{t}{\alpha}\right)^{A-1}\left(1 - \frac{t}{\beta}\right)^{B-1}$$

$$\cdot\left[\alpha B \int_0^1 \left(1 - v\frac{t}{\alpha}\right)^{-A}\left(1 - v\frac{t}{\beta}\right)^{-B-1} v^c\, dv\right.$$

$$\left.+ \beta A \int_0^1 \left(1 - v\frac{t}{\beta}\right)^{-B}\left(1 - v\frac{t}{\alpha}\right)^{-A-1} v^c\, dv\right]. \qquad (2.13)$$

Using (2.12) and (2.13) we obtain

$$Q_{l+kn}^{(\lambda)}(\cos\theta, 0, 0, c, k) = U_l(\cos\theta)\, P_n^{(\lambda+1)}(\cos k\theta, 0, 0, c, 1)$$
$$+ U_{k-l-2}(\cos\theta)\, P_{n-1}^{(\lambda+1)}(\cos k\theta, 0, 0, c, 1)$$

which generalizes the corresponding relation of Ismail [12, (2.28)].

3. Asymptotic Expansion of $Q_n^{(\lambda, r)}(x)$

We first determine the asymptotic behavior of $Q^{(\lambda, r)}(x, t)$ and their corresponding numerator polynomials. The tool used is the asymptotic method of Darboux. This method essentially states that if $f(z)=\sum_{n=0}^{\infty} f_n z^n$, $g(z)=\sum_{n=0}^{\infty} g_n z^n$ are analytic functions in $|z|<r$ and $f(z)-g(z)$ is continuous on $|z|=r$ then $f_n-g_n=o(r^{-n})$. This version of Darboux's method is basically the Riemann–Lebesgue Lemma.

Recall that the sieved Pollaczek polynomials of the second kind $B_n^\lambda(x)$ satisfy the limiting relation [7, (3.8)]

$$B_n^\lambda(x) \sim k^B \frac{\alpha}{\alpha-\beta}\left(1-\frac{\beta^k}{\alpha^k}\right)^A \frac{\beta^{-n}}{\Gamma(-B+1)} n^{-B} \quad \text{as} \quad n\to\infty \tag{3.1}$$

for $x\in\mathscr{C}/[-1,1]$.

THEOREM 3. *For fixed x in the cut plane $\mathscr{C}/[-1,1]$, we have*

$$\begin{aligned} Q_n^{(\lambda,r)}(x) \sim B_n^\lambda(x) \Big\{ & (\beta^{k+r}-\beta^{k-r})\,\beta^{k-r}A \\ & \cdot\int_0^1 \left(1-u\frac{\beta^k}{\alpha^k}\right)^{-A-1} (1-u)^{-B} u^c\,du \\ & + c\int_0^1 \left(1-u\frac{\beta^k}{\alpha^k}\right)^{-A} (1-u)^{-B} u^{c-1}\,du \Big\} \quad \textit{as} \quad n\to\infty. \end{aligned} \tag{3.2}$$

Proof. It is clear from (2.11) that the generating function $Q^{(\lambda,r)}(x,t)$ is analytic in t for $|t|<|\beta(x)|<|\alpha(x)|$ and has the same algebraic singularity as that of $B^\lambda(x,t)$ which is of order $-B(x)+1$ at $\beta(x)$. Since $\alpha\beta=1$ we shall sometimes write β^2 for β/α. Hence Darboux's method yields (3.2).

Let $\{Q_n^{*(\lambda,r)}(x)\}$ be the numerator polynomials associated with $\{Q_n^{(\lambda,r)}(x)\}$. In order to find the asymptotic expansion of $Q_n^{(\lambda,r)}(x)$ we first find the relationship between the these two sets of polynomials. For that purpose we find the corresponding relation between the unsieved polynomial sets $\{F_n^{*(\gamma)}(x)\}$ and $\{F_n^{(\gamma)}(x)\}$. The former satisfies the same recurrence relation as the latter, i.e., (2.4) with the initial conditions

$$F_0^{*(\gamma)}(x)=0, \qquad F_1^{*(\gamma)}(x)=\frac{2(1-\gamma U\Delta)}{1-\gamma q}.$$

It is now easy to verify that

$$F_n^{*(\gamma)}(x)=\frac{2(1-\gamma U\Delta)}{1-q\gamma} F_{n-1}^{q\gamma},$$

$$F_0^{*(\gamma)}(x)=0.$$

We now perform the sieving process (2.3) to get

$$Q_n^{*(\lambda,r)}(x)=\begin{cases}2Q_{n-1}^{(\lambda,r+1)}(x), & n\geqslant 1, r+1\neq k,\\ 2\left(\dfrac{\lambda+a+c+1}{c+1}\right)Q_{n-1}^{(\lambda,k)}(x), & r+1=k,\end{cases}\tag{3.3}$$

where $Q_{n-1}^{(\lambda,k)}(x)$ must be interpreted as $Q_{n-1}^{(\lambda,0)}(x)$ but with c being replaced by $c+1$ (see (2.3)).

Thus the asymptotic expansion for $Q_n^{*(\lambda,r)}(x)$ follows from (3.1) and (3.2).

Let K be the set $(x-\delta,\ x+\delta)\times(0,\delta)$, where $\delta>0$ is such that $\Re(B(z))<1$. For each $z\in K$ we can now determine the continued fraction

$$\chi(z)=\lim_{n\to\infty}\frac{Q_n^{*(\lambda,r)}(z)}{Q_n^{(\lambda,r)}(z)}=\frac{N}{D},\qquad r\neq k-1,\tag{3.4}$$

where

$$\begin{aligned}N=2\beta\Bigg\{&(\beta^{k+r+1}-\beta^{k-r-1})\,\beta^{k-r-1}A\\ &\times\int_0^{1]}\left(1-u\frac{\beta^k}{\alpha^k}\right)^{-A-1}(1-u)^{-B}u^c\,du\\ &+c\int_0^{1]}\left(1-u\frac{\beta^k}{\alpha^k}\right)^{-A}(1-u)^{-B}u^{c-1}\,du\Bigg\}\end{aligned}$$

$$\begin{aligned}D=&(\beta^{k+r}-\beta^{k-r})\,\beta^{k-r}A\\ &\cdot\int_0^{1]}\left(1-u\frac{\beta^k}{\alpha^k}\right)^{-A-1}(1-u)^{-B}u^c\,du\\ &+c\int_0^{1]}\left(1-u\frac{\beta^k}{\alpha^k}\right)^{-A}(1-u)^{-B}u^{c-1}\,du.\end{aligned}$$

When $r=k-1$ formula (3.4) still holds with the same value for D but N is now given by

$$N=2\beta(\lambda+a+c+1)\int_0^{1]}u^c\left(1-u\frac{\beta^k}{\alpha^k}\right)^{-A}(1-u)^{-B}\,du.$$

The continued fraction $\chi(z)$ is the Stieltjes transform of the distribution function, $d\phi(x)$, of $\{Q_n^{(\lambda,r)}(x)\}$, that is,

$$\chi(z)=\int_{-\infty}^{\infty}\frac{d\phi(t)}{z-t},\qquad z\notin\operatorname{supp}(d\phi).\tag{3.4a}$$

The left had side of (3.4), $\chi(z)$, fails to be analytic only on the support of $d\phi(x)$ where $\phi(x)$ is the distribution function for $\{Q_n^{(\lambda,r)}(x)\}$. The right hand side, on the other hand, is analytic on K and thus may be continued analytically to $\Omega=\{z\notin[-1,1] \text{ and } \Re\,(B(z))\neq 1,2,3,...\}$.

Recall that

$$\begin{aligned}
&\int_0^{1} (1-\beta^{2k}u)^{-A}(1-u)^{-B}u^{c-1}\,du\\
&\quad=\sum_{n\geqslant 0}\frac{(A)_n}{n!}\beta^{2nk}\int_0^{1}(1-u)^{-B}u^{n+c-1}\,du\\
&\quad=\frac{\Gamma(c)\,\Gamma(-B+1)}{\Gamma(-B+c+1)}\,{}_2F_1\left(\begin{matrix}A,c;\beta^{2k}\\ -B+c+1\end{matrix}\right).
\end{aligned}\tag{3.5}$$

THEOREM 4. *For $r\neq k-1$ and $z\in\Omega$ the continued fraction $\chi(z)$ whose denominators are $Q_n^{(\lambda,r)}(x)$ is given by*

$$\chi(z)=\frac{\left(\begin{aligned}&2\beta A\beta^{2k}(1-\alpha^{2r+2})\,{}_2F_1\left(\begin{matrix}A+1,c+1;\beta^{2k}\\ -B+c+2\end{matrix}\right)\\ &\qquad+2\beta(-B+c+1)\,{}_2F_1\left(\begin{matrix}A,c;\beta^{2k}\\ -B+c+1\end{matrix}\right)\end{aligned}\right)}{\left(\begin{aligned}&A\beta^{2k}(1-\alpha^{2r})\,{}_2F_1\left(\begin{matrix}A+1,c+1;\beta^{2k}\\ -B+c+2\end{matrix}\right)\\ &\qquad+(-B+c+1)\,{}_2F_1\left(\begin{matrix}A,c;\beta^{2k}\\ -B+c+1\end{matrix}\right)\end{aligned}\right)}.\tag{3.6}$$

Proof. Use (3.4), (3.5), and the defining formulas for N and D.

We remark that the right hand side of formula (3.6) has $-B+c+1=0$, -1, -2, ... as removable singularities. These singularities can be removed by dividing the numerator and the denominator by $\Gamma(-B+c+2)$.

We also have the companion theorem.

THEOREM 5. *For $r=k-1$ and $z\in\Omega$ the continued fraction $\chi(z)$ whose denominators are $Q_n^{(\lambda,r)}(x)$ is given by*

$$\chi(z)=\frac{2\beta(\lambda+a+c+1)\,{}_2F_1\left(\begin{matrix}A,c+1;\beta^{2k}\\ -B+c+2\end{matrix}\right)}{A\beta^{2k}(1-\alpha^{2r})\,{}_2F_1\left(\begin{matrix}A+1,c+1;\beta^{2k}\\ -B+c+2\end{matrix}\right)+(-B+c+1)\,{}_2F_1\left(\begin{matrix}A,c;\beta^{2k}\\ -B+c+1\end{matrix}\right)}.\tag{3.7}$$

4. Orthogonality Relation

The purpose of this section is to compute the absolutely continuous component of the distribution function, $\phi(x)$, with respect to which $\{Q_n^{(\lambda,r)}(x)\}$ is orthogonal. More precisely we shall determine $d\phi(x)/dx$. This component is clearly supported in $[-1, 1]$.

Let $x \in [-1, 1]$, then the discussion in [7, p. 1185] shows that we have $|\beta(x)| = |\alpha(x)|$ in this interval. In addition we assume that $x \neq \xi_j$ where $\xi_j = \cos(\pi j/k)$, $j = 1, 2, ..., k-1$. This last restriction makes

$$1 - \Re\left\{u\left(\frac{\beta(x)}{\alpha(x)}\right)^k\right\} > 0, \qquad 0 \leqslant u \leqslant 1. \tag{4.1}$$

Thus we may choose $\gamma > 0$ and $\delta > 0$ so that

$$1 - \Re\left\{u\left(\frac{\beta(z)}{\alpha(z)}\right)^k\right\} \geqslant \gamma, \qquad 0 \leqslant u \leqslant 1 - \delta \tag{4.2}$$

for all z in the compact set $K_x = [x-\delta, x+\delta] \times [0, \delta]$.

In order to compute $\phi'(x)$ for $-1 < x < 1$ we need to evaluate

$$\hat{\chi}(t) = \lim_{\varepsilon \to 0+} (\chi(t - i\varepsilon) - \chi(t + i\varepsilon)), \qquad t \in [x-\delta, x+\delta]. \tag{4.3}$$

The inequalities (4.1) and (4.2) ensure the uniform convergence of (4.3). Thus the Perron–Stieltjes inversion formula gives

$$\phi'(x) = \frac{1}{2\pi i}\hat{\chi}(x) \qquad \text{for} \quad x \in (\xi_j, \xi_{j-1}), \tag{4.4}$$

where in (4.4), $j = 1, 2, ..., k$, $\xi_0 := 1$, $\xi_k := -1$. Using (3.6) we get

$$\begin{aligned}\hat{\chi}(x) = \Bigg\{ 2(\alpha - \beta)\Bigg[&\alpha^{2k}B(-B+c+1)\,{}_2F_1\left(\begin{matrix}B+1, c+1; \alpha^{2k}\\ -A+c+2\end{matrix}\right)\\ &\times {}_2F_1\left(\begin{matrix}A, c; \beta^{2k}\\ -B+c+1\end{matrix}\right) + \beta^{2k}A(-A+c+1)\\ &\times {}_2F_1\left(\begin{matrix}A+1, c+1; \beta^{2k}\\ -B+c+2\end{matrix}\right){}_2F_1\left(\begin{matrix}B, c; \alpha^{2k}\\ -A+c+1\end{matrix}\right)\end{aligned}$$

$$+(-A+c+1)(-B+c+1)$$
$$\times {}_2F_1\left(\begin{matrix} B, c; \alpha^{2k} \\ -A+c+1 \end{matrix}\right) {}_2F_1\left(\begin{matrix} A, c; \beta^{2k} \\ -B+c+1 \end{matrix}\right)\Bigg]\Bigg\}$$
$$\times \Bigg| B\alpha^{2k}(1-\beta^{2r})\, {}_2F_1\left(\begin{matrix} B+1, c+1; \alpha^{2k} \\ -A+c+2 \end{matrix}\right)$$
$$+(-A+c+1)\, {}_2F_1\left(\begin{matrix} B, c; \alpha^{2k} \\ -A+c+1 \end{matrix}\right)\Bigg|^{-2}. \tag{4.5}$$

In the numerator we use Kummer's solutions of the hypergeometric differential equation [10, p. 105]. More specifically put $y=\beta^{2k}$ and

$$u_1 = {}_2F_1\left(\begin{matrix} c, A; y \\ -B+c+1 \end{matrix}\right), \qquad u_3 = (-y)^{-c}\, {}_2F_1\left(\begin{matrix} c, B; y^{-1} \\ -A+c+1 \end{matrix}\right).$$

Then the numerator of (4.5) becomes

$$\frac{2(\alpha-\beta)(-B+c+1)(-A+c+1)(-y)^{c+1}}{c} W(y), \tag{4.6}$$

where $W(y)=u_1(y)\,u_3'(y)-u_3(y)\,u_1'(y)$, which satisfies the differential equation

$$y(1-y)\,W'(y)+[(-B+c+1)-(A+c+1)\,y]\,W(y)=0. \tag{4.7}$$

Since $A+B=-2\lambda$ the general solution of (4.7) is

$$W(y)=E(1-y)^{2\lambda}\,y^{B-c-1}=E(1-\beta^{2k})^{2\lambda}\,(\alpha^{2k})^{-B+c+1}. \tag{4.8}$$

To evaluate the constant E we use the following analytic continuation of the hypergeometric series [10, p. 108, (2.10)]

$$u_1(y)=C_1u_3(y)+C_2u_4(y), \tag{4.9}$$

where

$$u_4(y)=(-y)^{-A}\, {}_2F_1\left(\begin{matrix} A, -2\lambda-c; 1/y \\ 1-c+A \end{matrix}\right),$$
$$C_1=\frac{\Gamma(1+c-B)\,\Gamma(A-c)}{\Gamma(-B+1)\,\Gamma(A)}, \quad C_2=\frac{\Gamma(-B+c+1)\,\Gamma(c-A)}{\Gamma(c)\,\Gamma(1+c+2\lambda)}.$$

Putting (4.9) in the definition of $W(y)$ and comparing the result with (4.8) we get

$$E=(-1)^{c+1-B}\frac{\Gamma(c+1-B)\,\Gamma(c+1-A)}{\Gamma(c)\,\Gamma(1+c+2\lambda)}. \tag{4.10}$$

Using these values in (4.5),

$$\phi'(x)=\frac{2\sqrt{1-x^2}\,|\Gamma(-A+c+2)|^2}{\pi\Gamma(c+1)\,\Gamma(c+1+2\lambda)}\,|(1-\beta^{2k})^{-A}|^2$$
$$\times\left|B\alpha^{2k}(1-\beta^{2r})\,{}_2F_1\binom{B+1,\,c+1;\,\alpha^{2k}}{-A+c+2}\right.$$
$$\left.+(-A+c+1)\,{}_2F_1\binom{B,\,c;\,\alpha^{2k}}{-A+c+1}\right|^{-2}. \qquad (4.11)$$

For $c=r=0$ this leads to the absolutely continuous component of the distribution for the Pollaczak polynomials

$$\phi'(x)=\frac{2\sqrt{1-x^2}}{\pi}\,\frac{\Gamma(-B+1)\,\Gamma(-A+1)}{\Gamma(2\lambda+1)}\,(1-\beta^{2k})^{-A}\,(1-\alpha^{2k})^{-B},$$

which gives formula [6, (5.11)].

If we recall that, for $-1\leqslant x=\cos\theta\leqslant 1$,

$$A=\bar{B}=-\lambda-\frac{ax+b}{\sqrt{1-x^2}}\,i=-\lambda-ih(\theta)$$

$$\alpha=\bar{\beta}=x+i\sqrt{1-x^2}=e^{i\theta}$$

then

$$(1-\alpha^{2k})^{-B}\,(1-\beta^{2k})^{-A}=(1-e^{-2ik\theta})^{-A}\,(1-e^{2ik\theta})^{-B}$$
$$=2^{2\lambda}\,|U_{k-1}(x)|^{2\lambda}\,(1-x^2)^{\lambda}\,e^{(2k\theta-\pi)h(\theta)}$$

so that (4.11) may be rewritten as

$$\phi'(x)=\frac{2^{2\lambda+1}(1-x^2)^{\lambda+1/2}\,|\Gamma(\lambda+c+2+ih(\theta))|^2}{\pi\Gamma(c+1)\,\Gamma(c+1+2\lambda)}$$
$$\times|U_{k-1}(x)|^{2\lambda}\exp\{(2k\theta-\pi)\,h(\theta)\}$$
$$\times\left|B\alpha^{2k}(1-\beta^{2r})\,{}_2F_1\binom{B+1,\,c+1;\,\alpha^{2k}}{-A+c+2}\right.$$
$$\left.+(-A+c+1)\,{}_2F_1\binom{B,\,c;\,\alpha^{2k}}{-A+c+1}\right|^{-2}. \qquad (4.12)$$

5. Remarks and Problems

An isolated point mass A_0 at $x = x_0$ contributes a simple pole to the singularities of $\chi(z)$. Furthermore, (3.4a) shows that the residue of $\chi(z)$ at $z = x_0$ is A_0. Therefore, by (3.6), the set of isolated points of the discrete spectrum coincides with the set of zeros of

$$A\beta^{2k}(1-\alpha^{2r})\,{}_2F_1\left(\begin{matrix} A+1, c+1; \beta^{2k} \\ 2+c-B \end{matrix}\right) + (1+c-B)\,{}_2F_1\left(\begin{matrix} A, c; \beta^{2k} \\ -B+1+c \end{matrix}\right). \quad (5.1)$$

Determining the latter (zero) set is not easy even in the special case $c = 0$, and $r > 0$, where finding the zeros of the expression (5.1) reduces to solving the transcendental equation

$${}_2F_1\left(\begin{matrix} A(x)+1, 1; \beta^{2k} \\ 2-B(x) \end{matrix}\right) = \frac{B(x)-1}{A(x)} \frac{\alpha^{2k}}{(1-\alpha^{2k})}. \quad (5.2)$$

In (5.2) we replaced A and B by $A(x)$ and $B(x)$ to exhibit their dependence on x. As a matter of fact α and β also depend on x,

$$\alpha = x \pm \sqrt{x^2-1}, \qquad \beta = x \mp \sqrt{x^2-1},$$

$$A(x) = -\lambda \pm \frac{ax+b}{\sqrt{x^2-1}}, \qquad B(x) = -\lambda \mp \frac{ax+b}{\sqrt{x^2-1}}.$$

Some things are known about the distribution of zeros [10] of the ${}_2F_1$'s in some special cases but nothing seems to be known about solutions of the transcendental equations like (5.2). The question of studying the distribution of zeros of hypergeometric and confluent hypergeometric functions needs to be reexamined and existing results need to be extended to cover solutions of equations like (5.2) or zeros of expressions like (5.1).

The zeros of $\{Q_n^{(\lambda,r)}(x)\}$ are differentiable functions of c for any fixed r. When $r = c = 0$ the spectrum of the distribution function is computed in some detail in [7]. Therefore when $|c|$ is small one would expect the discrete spectrum to continue to behave like the discrete spectrum when $c = 0$.

Let $\{p_n(x)\}$ be a sequence of polynomials generated by

$$\begin{aligned} &xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x), \qquad n > 0, \\ &p_{-1}(x) = 0, \qquad p_0(x) = 1, \end{aligned} \quad (5.3)$$

with a_n and b_n real, $a_n \neq 0$ for all $n \geqslant 0$. It is well known that $\{p_n(x)\}$ are orthogonal with respect to a positive measure whose moments are finite and whose support is infinite if and ony if the positivity conditions

$$a_{n-1}c_n > 0 \qquad \text{for all} \quad n > 0 \quad (5.4)$$

hold. In the case of (2.6) the above positivity conditions will be satisfied if

$$c+1>0, \qquad 2\lambda+1+c>0, \qquad \lambda+a+c+1>0. \tag{5.5}$$

The condition $c+1>0$ is very natural since c is the association parameter and one would undoubtedly encounter indeterminacies if c were allowed to become a negative integer. There may be cases of orthogonality if $c<-1$, $c \neq -2, -3, \ldots$ but the analysis in these cases will be very tricky. If we assume $c>-1$ then the remaining inequalities in (5.5) hold if and only if the positivity conditions hold; that is, if and only if the polynomials $Q_n^{(\lambda, r)}(x)$ are orthogonal.

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